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Coulomb scattering for scalar field in Schrödinger picture

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ABSTRACT

The scattering of a charged scalar field on Coulomb potential on de Sitter space–time is studied using the solution of the free Klein–Gordon equation. We find that the scattering amplitude is independent of the choice of the picture and in addition the total energy is conserved in the scattering process.

1. Introduction

Recently a new time-evolution picture was defined in charts with spatially flat Robertson–Walker metrics, under the name of Schrödinger picture [1]. Using the advantage offered by this picture in [2] was found a new set of solutions for Klein–Gordon equation on de Sitter space–time which behaves as polarized plane waves with a given energy. In the present Letter we will exploit this, for calculate the Coulomb scattering for a charged scalar field in de Sitter expanding universe.

The Letter is organized as follows. In Section 2, we present a short review of the Schrödinger picture introduced in [1] and we write the form of the solutions for Klein–Gordon equation. In Section 3 we define the lowest order contribution for scalar field in the potential $A^{\hat{\mu}}$ in the new Schrödinger picture and then we calculate the scattering amplitude, showing that the total energy is conserved. Our conclusions are summarized in Section 4. We use elsewhere natural units, i.e., $\hbar = c = 1$.

2. Plane waves with a given energy

Let us take the local chart with Cartesian coordinates of a flat Robertson–Walker manifold, in which the line element reads:

$$ds^2 = dt^2 - \alpha(t)^2\, d\vec{x}^2, \tag{1}$$

where α is an arbitrary function. The scalar field ϕ of mass m satisfy the free Klein–Gordon equation. If $\phi(x)$ is the scalar field in the natural picture then the scalar field of the Schrödinger picture can be written as $\phi_S(x) = W(x)\phi(x)$ [2], where $W(x)$ is the operator of time dependent dilatations [1],

$$W(x) = \exp\bigl[-\ln\bigl(\alpha(t)\bigr)(\vec{x}\cdot\vec{\partial})\bigr], \tag{2}$$

which has the remarkable property:

$$W(x)^+ = \sqrt{-g(t)}\,W(x)^{-1}. \tag{3}$$

Now taking in Eq. (1) $\alpha(t) = e^{\omega t}$ one obtains the de Sitter metric which is the case of interest here. We start with the solutions of the free Klein–Gordon equations written on de Sitter space–time which was obtained in [2] using the Schrödinger picture. The fundamental solutions in the Schrödinger picture $f^S_{E,\mathbf{n}}$ of positive frequencies, with energy E and momentum direction $\mathbf{n}$ resulted from [2] have the integral representation:

$$f^S_{E,\mathbf{n}}(x) = \frac{1}{2}\sqrt{\frac{\omega}{2}}\frac{e^{-iEt}e^{-\pi k/2}}{(2\pi)^{3/2}}\int_0^\infty ds\,\sqrt{s}H^{(1)}_{ik}(s)e^{i\mathbf{p}\cdot\mathbf{x}-i\epsilon\ln s}, \tag{4}$$

where $s = p/\omega$ and $H^{(1)}_\mu(z)$ is a Hankel function of first kind. Then the solution in the natural picture obtained in [2] read:

$$f_{E,\mathbf{n}}(x) = \frac{1}{2}\sqrt{\frac{\omega}{2}}\frac{e^{-iEt}e^{-\pi k/2}}{(2\pi)^{3/2}}\int_0^\infty ds\,\sqrt{s}H^{(1)}_{ik}(s)e^{i\mathbf{p}\cdot\mathbf{x}_t-i\epsilon\ln s}, \tag{5}$$

with $\mathbf{x}_t = e^{\omega t}\mathbf{x}$.

These solutions satisfy the orthonormalization relations:

$$\begin{aligned} &i\int d^3x\,(-g)^{1/2} f^*_{E,\mathbf{n}}(x)\overleftrightarrow{\partial_t} f_{E',\mathbf{n}'}(x) \\ &\quad= -i\int d^3x\,(-g)^{1/2} f_{E,\mathbf{n}}(x)\overleftrightarrow{\partial_t} f^*_{E',\mathbf{n}'}(x) = \delta(E-E')\delta^2(\mathbf{n}-\mathbf{n}'), \\ &i\int d^3x\,(-g)^{1/2} f_{E,\mathbf{n}}(x)\overleftrightarrow{\partial_t} f_{E',\mathbf{n}'}(x) = 0, \end{aligned} \tag{6}$$

where the integration extends on an arbitrary hypersurface $t =$ const and $(-g)^{1/2} = e^{3\omega t}$, and the completeness condition

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$$i\int_0^\infty dE\int_{S^2} d\Omega_n\{[f_{E,\mathbf{n}}(t,\mathbf{x})]^*\overset{\leftrightarrow}{\partial_t} f_{E,\mathbf{n}}(t,\mathbf{x}')\} = e^{-3\omega t}\delta^3(\mathbf{x}-\mathbf{x}'). \tag{7}$$

Since one study of the scattering in the energy basis was not done, we will focus in this Letter on this problem. Our strategy is to obtain the scattering amplitude in the Schrödinger picture and then to translate our calculations in the natural picture.

3. Scattering amplitude and cross section

The scattering theory in de Sitter space–time can be constructed using the methods from flat space case. Then in analogy with Minkowski case [3] the scattering amplitude for a charged scalar field in the first order of the perturbation theory can be defined as follows:

$$A_{i\to f} = -e\int \sqrt{-g(x)}\,[f_f^*(x)\overset{\leftrightarrow}{\partial_\mu} f_i(x)]A^\mu(x)\,d^4x, \tag{8}$$

this being just the scattering amplitude in the natural picture. It is then a simple calculations to obtain the scattering amplitude in the new Schrödinger picture using Eq. (3), we obtain:

$$A^S_{i\to f} = -e\int [f_f^{S*}(x)\overset{\leftrightarrow}{\partial_\mu} f_i^S(x)]A_S^\mu(x)\,d^4x. \tag{9}$$

If we write for the Coulomb potential on de Sitter space $A^{\hat{0}}(x) = \frac{Ze}{|\vec{x}|}e^{-\omega t}$, then in the new Schrödinger picture this will become:

$$A_S^{\hat{0}}(x) = \frac{Ze}{|\vec{x}|}. \tag{10}$$

Our aim here is to calculate the amplitude of Coulomb scattering for a charged scalar field using the definition (9) in which we replace the solutions (4) and the potential (10). Let us starting with the waves freely propagating in the *in* and *out* sectors, $f^S_{E_i,\mathbf{n}}(x)$ and $f^S_{E_f,\mathbf{n}}(x)$, assuming that the both of them are of positive frequency. If we replace these quantities and perform the bilateral derivative, we observe that the spatial and temporal integrals have the same form as in Minkowski theory, and we obtain for our amplitude:

$$A^S_{i\to f} = \frac{i\alpha Z\omega(E_f+E_i)}{8\pi|\vec{p}_f-\vec{p}_i|^2}\delta(E_f-E_i)\left[\int_0^\infty ds_f\, s_f^{1/2+iE_f/\omega}H^{(2)}_{ik}(s_f)\right.$$
$$\left.\times\int_0^\infty ds_i\, s_i^{1/2-iE_i/\omega}H^{(1)}_{ik}(s_i)\right], \tag{11}$$

where $\alpha = e^2$.

The form of integrals that help us to obtain the final form of scattering amplitude are given in Appendix A, here we give just the final result in terms of gamma Euler functions and delta Dirac function:

$$A^S_{i\to f} = \frac{i\alpha Z\omega(E_f+E_i)}{4\pi|\vec{p}_f-\vec{p}_i|^2}\delta(E_f-E_i)[g_k(E_f)g^*_{-k}(E_i)]. \tag{12}$$

For simplification in (12), we introduce the following notation:

$$g_k(E) = 2^{iE/\omega}\frac{\Gamma(\frac{3}{4}+\frac{ik}{2}+\frac{iE}{2\omega})}{\Gamma(\frac{1}{4}+\frac{ik}{2}-\frac{iE}{2\omega})} - \frac{i2^{iE/\omega}}{\pi}\sin\left(\frac{\pi}{4}-\frac{ik\pi}{2}+\frac{iE\pi}{2\omega}\right)$$
$$\times\Gamma\left(\frac{3}{4}+\frac{ik}{2}+\frac{iE}{2\omega}\right)\Gamma\left(\frac{3}{4}-\frac{ik}{2}+\frac{iE}{2\omega}\right), \tag{13}$$

and $g_{-k}(E)$ is obtained when $k\to -k$ in (13).

Let us see now how the above results can be translated in the natural picture. The definition of the scattering amplitude in the natural picture is given in (8) and the form of Coulomb potential in the natural picture is $A^{\hat{0}}(x) = \frac{Ze}{|\vec{x}|}e^{-\omega t}$. This time in our calculations we use the solutions of free Klein–Gordon equation written in natural picture (5). After a little calculation one can observe that the result is the same as in (12). This can be checked passing to a new variable of integration $y = xe^{\omega t}$ when we solve the spatial integrals. It means that the result for the scattering amplitude is independent of the choice of picture in which one works.

Now let us make some comments about our scattering amplitude (12). We obtain that the energy is conserved in the scattering process as in Minkowski case. This result is expected because the form of the external field (10) allows us to consider that the scattering process take place in a constant field. But it is known that the energy of a system scattered on a constant field is conserved [5] (this does not mean that the momentum is also conserved), as we obtained here.

As we know the cross section can be defined in this case as [4]:

$$d\sigma = \frac{dP}{dt}\frac{1}{j}, \tag{14}$$

where $\frac{dP}{dt}$ is the transition probability in unit of time and j is the incident flux.

The incident flux can be obtained using the definition of scalar current of particles. In the present case the incident flux calculated in the Schrödinger picture is:

$$j = -i[f^{S*}_{E_i\mathbf{n}}(x)\overset{\leftrightarrow}{\partial_i} f^S_{E_i\mathbf{n}}(x)] = \frac{p_i\omega}{2(2\pi)^3}g_k(E_i)g^*_{-k}(E_i). \tag{15}$$

The evaluation of the probability in unit $\frac{dP_l}{dt} = \frac{d|A^S_{i\to f}|^2}{dt}\frac{d^3p_f}{(2\pi)^3}$ of time is immediate and we obtain:

$$\frac{dP_l}{dt} = \frac{(\alpha Z)^2\omega^2(E_f+E_i)^2}{32\pi^3|\vec{p}_f-\vec{p}_i|^4}\delta(E_f-E_i)|g_k(E_f)|^2|g_{-k}(E_i)|^2. \tag{16}$$

The differential cross section will be obtained after we replace (15) and (16) in (14):

$$d\sigma = \frac{\omega(\alpha Z)^2\delta(E_f-E_i)(E_f+E_i)^2}{16p_i\pi^3|\vec{p}_f-\vec{p}_i|^4}\frac{[|g_k(E_f)|^2|g_{-k}(E_i)|^2]}{g_k(E_i)g^*_{-k}(E_i)}d^3p_f. \tag{17}$$

As we know in Minkowski case the factor with $\delta(E_f-E_i)$ was eliminated after performing the integral with respect to the final momentum. In the present case an integration with respect final momentum is required for obtaining explicitly the differential cross section. However it is important to say that because we do not have a relation that connect the momentum and energy as in Minkowski case the distributional factor cannot be eliminated when we perform the integration with respect to final momentum in (17).

We observe that our cross sections have a complicated dependence of energy. This dependence of energy was obtained after the integration with respect to $s = p/\omega$, which means that this dependence translated in physical terms means that our cross section is dependent of the form of the incident wave which is unusual.

4. Conclusion

We examined in this Letter the Coulomb scattering of a charged massive scalar field on de Sitter space–time using the plane wave solutions of free Klein–Gordon equation. The solutions that we use were obtained in Schrödinger and natural pictures and behave as plane waves with a given energy. An important result is the fact that the scattering amplitude is independent of the picture in which we work.

We found that the scattering amplitude and the cross sections depend on the expansion factor as ω. The result obtained here shows that the amplitude and the cross section depends on the form of the incident wave. Needless to say that this consequences is the result of the lost translational invariance with respect to time in de Sitter space–time. Also we found that the total energy is conserved in the scattering process and, in addition, terms that could break the energy conservation are absent since the scattering was considered in a constant field of the form (10).

For further investigations it will be interesting to develop the entire scattering theory in Schrödinger picture. This will require a perturbation theory and one reduction formalism for the scalar field, which must be developed in the new Schrödinger picture. For that one must use the form of the Klein–Gordon equation in the Schrödinger picture [2] and the fundamental solutions of positive/negative frequencies.

Appendix A

For obtaining the scattering amplitude we need the formula [6]:

$$H_\nu^{(1,2)}(z) = J_\nu(z) \pm iY_\nu(z), \tag{18}$$

which replaced in our amplitude led to integrals whose general form is [6]:

$$\int_0^\infty dz\, z^\mu J_\nu(z) = 2^\mu \frac{\Gamma(\frac{\mu+\nu+1}{2})}{\Gamma(\frac{\nu-\mu+1}{2})},$$

$$Re(\mu+\nu) > -1, \qquad Re(\mu) < \frac{1}{2} \tag{19}$$

and

$$\int_0^\infty dz\, z^\mu Y_\nu(z) = \frac{2^\mu}{\pi}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)\sin\frac{\pi}{2}(\mu-\nu),$$

$$Re(\mu\pm\nu) > -1, \qquad Re(\mu) < \frac{1}{2}. \tag{20}$$

Now setting $z = p/\omega$ and $\nu = ik$ and $\mu = 1/2 \pm iE/\omega$ one can see that our result (12) is correct. Also the above integrals help us to obtain the incident flux.

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